THE ROTATION CLASS OF A FLOW

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Abstract. Generalizing a construction of A. Weil, we introduce a topological invariant for flows on compact, connected, finite dimensional, abelian, topological groups. We calculate this invariant for some examples and compare the invariant with other flow invariants.

1. Introduction

For our purposes a *flow* is a continuous group action of $(\mathbf{R},+)$. We consider the flow ϕ on X and the flow ψ on Y to be *topologically equivalent* when there is a homeomorphism $h : X \to Y$ which takes orbits of ϕ onto orbits of ψ in such a way that the orientation of orbits is preserved, in which case we write $h : \phi \overset{\text{top}}{\approx} \psi$. Following the lead of A. Weil as outlined in [W1][W2], we introduce the *rotation class* of a flow: an invariant of topological equivalence for flows on finite dimensional, compact, connected, abelian, topological groups – hereafter referred to as $n-$solenoids, see definition 1 or [C2]. A modern account of Weil's torus invariant and its application to toral flows may be found in [AZ],[ABZ, Chapter 6]. For each $n-$ solenoid we shall describe a covering space and a compactification of the covering space, the *compactification of perspective*. The remainder of this compactification is homeomorphic to S^{n-1}. Informally stated, we shall show that if ϕ and ψ are topologically equivalent flows on the $n-$solenoid $\sum_{\overline{M}}$ and if the lifted flow of ϕ has an orbit which has an $\omega-$limit point in the remainder of the compactification of perspective, then any corresponding orbit in the lifted flow of ψ will have a corresponding $\omega-$limit point in the remainder of the compactification. The group structure of $\sum_{\overline{M}}$ determines which points in the remainder may correspond.

We compare this invariant with other flow invariants from [C1] and [MZ] and calculate these invariant in some examples. In the course of doing this we provide a general technique for calculating the exponent

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group as introduced in [C1]. It will then be apparent that the rotation class is well suited for investigating topological equivalence but that it is not as well suited as other invariants for determining ergodic properties of the associated flow.

2. Background and the Compactification of Perspective

Definition 1. *For a fixed n and a sequence $\overline{M} = (M_1, M_2, ...)$ of $n \times n$ matrices M_i with integer entries and non-zero determinants, we define the topological group $\sum_{\overline{M}}$ with identity $e_{\overline{M}}$ to be the inverse limit of the inverse sequence $\{\mathbf{X}_j, f_j^i\}$, where $\mathbf{X}_j = \mathbf{T}^n$ for all $j \in \mathbf{N}$ and f_j^{j+1} is the topological epimorphism represented by the matrix M_j; $f_j^{j+1} \circ p^n = p^n \circ M_j$.*

$$\sum\nolimits_{\overline{M}} \overset{def}{=} \left\{ \langle x_j \rangle_{j=1}^{\infty} \in \prod_{j=1}^{\infty} \mathbf{T}^n | f_j^{j+1}(x_{j+1}) = x_j \text{ for all } j \in \mathbf{N} \right\},$$

*and we define such an inverse limit $\sum_{\overline{M}}$ to be an n-*solenoid.

We shall only consider the case that $n \in \{2, 3, ...\}$. As in [C2] we have the fibration with unique path lifting

$$\begin{aligned} \pi_{\overline{M}} &: \mathbf{R}^n \to \sum\nolimits_{\overline{M}} \\ \pi_{\overline{M}}(s) &= \left(\pi^n(s), \pi^n\left(M_1^{-1}(s)\right), ..., \pi^n\left(M_n^{-1} \circ \cdots \circ M_1^{-1}(s)\right), ... \right), \end{aligned}$$

where $\pi^n : \mathbf{R}^n \to \mathbf{T}^n = (\mathbf{R}/\mathbf{Z})^n$ is the quotient covering map. If p_i denotes the projection of $\sum_{\overline{M}}$ onto the i^{th} $\mathbf{T}^n$ factor, then $p_1^{-1}(e)$ is a Cantor set or a finite discrete space. We may then form the following covering of $\sum_{\overline{M}}$.

Definition 2.

$$E_{\overline{M}} \overset{def}{=} \mathbf{R}^n \times p_1^{-1}(e)$$

and

$$\Pi_{\overline{M}} : E_{\overline{M}} \to \sum\nolimits_{\overline{M}}$$

is the covering map given by

$$\Pi_{\overline{M}}(s, x) = \pi_{\overline{M}}(s) + x.$$

See [M] for a similarly defined covering map.

With $h_n : \mathbf{R}^n \to \mathbf{D}^n = \{x \in \mathbf{R}^n | \, \|x\| < 1\}$ denoting the homeomorphism

$$x = (x_1, .., x_n) \mapsto \frac{1}{\sqrt{1 + x_1^2 + \cdots + x_n^2}} \cdot x$$

and identifying a point x of $\mathbf{R}^n$ with its h_n image in $\mathbf{R}^n$, we may consider $\overline{\mathbf{D}^n} = \{x \in \mathbf{R}^n | \, \|x\| \leq 1\}$ to be a compactification of $\mathbf{R}^n$. Similarly we may consider $\overline{\mathbf{D}^n} \times p_1^{-1}(e)$ to be a compactification of $E_{\overline{M}}$ via $h_n \times id_{p_1^{-1}(e)}$. With $S^{n-1} = \{x \in \mathbf{R}^n | \, \|x\| = 1\}$, we form the equivalence relation $\approx_{\overline{M}}$ on $\overline{\mathbf{D}^n} \times p_1^{-1}(e)$ defined by

$$[(s,x) \approx_{\overline{M}} (s',x')] \iff \left[s = s' \in S^{n-1}\right].$$

We then have the quotient mapping

$$q_{\overline{M}} : \overline{\mathbf{D}^n} \times p_1^{-1}(e) \to \left(\overline{\mathbf{D}^n} \times p_1^{-1}(e)\right) / \approx_{\overline{M}},$$

the image of which is also a compactification of $E_{\overline{M}}$, *the compactification of perspective*. The remainder of this compactification,

$$R_{\overline{M}} \overset{\text{def}}{=} \left[\left(\overline{\mathbf{D}^n} \times p_1^{-1}(e)\right) / \approx_{\overline{M}}\right] - \left[q_{\overline{M}} \circ \left(h_n \times id_{p_1^{-1}(e)}\right)(E_{\overline{M}})\right]$$

is canonically homeomorphic to S^{n-1}

$$[(t,x)]_{\approx_{\overline{M}}} \rightsquigarrow t,$$

where $[(t,x)]_{\approx_{\overline{M}}}$ denotes the $\approx_{\overline{M}}$-class of a point in $R_{\overline{M}}$. Hereafter we identify $R_{\overline{M}}$ with S^{n-1} as above.

In [C2] the linear flows on $\sum_{\overline{M}}$

$$\mathcal{F}_{\overline{M}} = \{\Phi^{\omega}_{\overline{M}} \mid \omega \in \mathbf{R}^n - \{\mathbf{0}\}\} \text{ given by}$$

$$\Phi^{\omega}_{\overline{M}} : \mathbf{R} \times \sum\nolimits_{\overline{M}} \to \sum\nolimits_{\overline{M}}; \; \Phi^{\omega}_{\overline{M}}(t,x) = \pi_{\overline{M}}(t \cdot \omega) + x$$

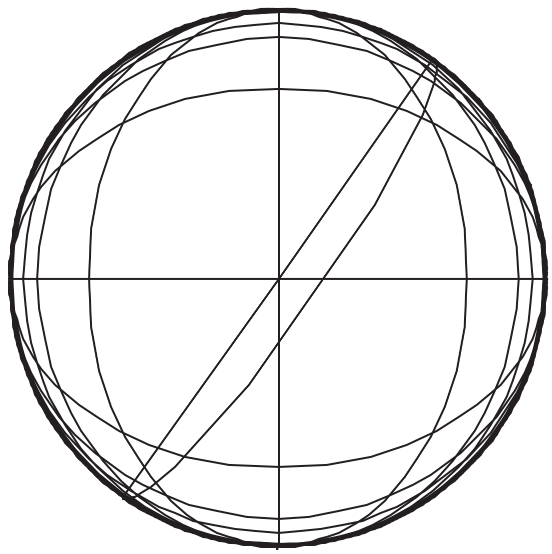
were introduced. Each linear flow $\Phi^{\omega}_{\overline{M}}$ lifts to the flow $\widetilde{\Phi^{\omega}_{\overline{M}}}$ on $E_{\overline{M}}$ given by

$$\widetilde{\Phi^{\omega}_{\overline{M}}}(t,(s,x)) = (t \cdot \omega + s, x).$$

Since $h_n(t \cdot \omega + s) \to \dfrac{\omega}{\|\omega\|}$ as $t \to \infty$, the image of any $\widetilde{\Phi^{\omega}_{\overline{M}}}$–orbit in the compactification of perspective tends asymptotically toward the same point $\dfrac{\omega}{\|\omega\|}$ in the remainder S^{n-1}. This leads naturally to the following.

Definition 3. *The point $x \in S^{n-1}$ is a* point of perspective *of the flow ϕ on $\sum_{\overline{M}}$ if and only if ϕ has an orbit which lifts to an orbit $\mathcal{O}$ in the compactification of perspective satisfying $\{\mathcal{O}(t_i)\}_{i \in \mathbf{N}} \to x$ for some sequence of real numbers $\{t_i\}_{i \in \mathbf{N}} \to \infty$.*

Thus, the linear flow $\Phi^{\omega}_{\overline{M}}$ has precisely one point of perspective: $\dfrac{\omega}{\|\omega\|}$.



The compactification of perspective of the torus, along with the integer coordinate grid and two orbits of an irrational flow.

3. The Rotation Class of a Flow

In [C2] the linear flows were classified up to topological equivalence. The automorphisms of $\sum_{\overline{M}}$ play a key role in this classification. If $\alpha : \sum_{\overline{M}} \to \sum_{\overline{M}}$ is an automorphism, it is shown in [C2] that α lifts to an automorphism $\mathbf{R}^n \to \mathbf{R}^n$, which may then be represented by a matrix $A(\alpha)$, as indicated in the following commutative diagram

$$\begin{array}{ccc} \mathbf{R}^n & \overset{A(\alpha)}{\to} & \mathbf{R}^n \\ {}^{\pi_{\overline{M}}}\downarrow & & \downarrow^{\pi_{\overline{M}}} \\ \sum_{\overline{M}} & \overset{\alpha}{\to} & \sum_{\overline{M}} \end{array} .$$

Any automorphism $A : \mathbf{R}^n \to \mathbf{R}^n$ induces the map $\widehat{A} : S^{n-1} \to S^{n-1}$

$$\widehat{A}(x) = \frac{A(x)}{\|A(x)\|}.$$

Translating the classification of linear flows into the language of perspective leads to the following notions.

Definition 4. *For $X \subset S^{n-1}$*

$$\overline{M}(X) \overset{def}{=} \cup \left\{ \widehat{A(\alpha)}(X) \,|\, \alpha \in Aut\left(\sum_{\overline{M}}\right) \right\},$$

and the rotation class *of a flow ϕ on $\sum_{\overline{M}}$ is given by*

$$\rho(\phi) \overset{def}{=} \overline{M}(\{\text{points of perspective of } \phi\}).$$

Note Since $x \mapsto -x$ is an automorphism of $\sum_{\overline{M}}$ for any $\overline{M}$, we could work in real projective $n-1$ space just as well as in S^{n-1}. When applied to the torus, $\rho(\phi)$ is called the rotation orbit of ϕ in [ABZ, Chapter 6].

The topological classification of linear flows given in [C2] may then be rephrased as follows:

$$\left[\Phi_{\overline{M}}^{\omega} \text{ is topologically equivalent to } \Phi_{\overline{M}}^{\omega'}\right] \iff \left[\rho\left(\Phi_{\overline{M}}^{\omega}\right) = \rho\left(\Phi_{\overline{M}}^{\omega'}\right)\right].$$

We now investigate how this can be generalized to other flows.

Lemma 5. *If ϕ and ψ are flows on $\sum_{\overline{M}}$ and $h : \phi \overset{top}{\approx} \psi$ with h homotopic to translation by some $\tau \in \sum_{\overline{M}}$, then*

$$\{\textit{points of perspective of } \phi\} = \{\textit{points of perspective of } \psi\}.$$

Proof Suppose $\omega \in S^{n-1}$ is a point of perspective of ϕ and that $\mathcal{O}$ is a lifted ϕ−orbit of $x_0 \in \sum_{\overline{M}}$ so that for some sequence of real numbers $\{t_i\}_{i\in\mathbf{N}} \to \infty$ $\{\mathcal{O}(t_i)\}_{i\in\mathbf{N}} \to \omega$ in the compactification of perspective. Generally as $t \to \infty$ the orbits of the lifted linear flow $\widetilde{\Phi_{\overline{M}}^{\omega}}$ approach ω asymptotically in the compactification of perspective. Let $\mathbf{R}^n \times \{c\}$ be the path component of $E_{\overline{M}}$ containing $\mathcal{O}(\mathbf{R})$, which is foliated by $\widetilde{\Phi_{\overline{M}}^{\omega}}$. With each $x \in \mathbf{R}^n \times \{c\}$ we associate the ray

$$A_x \overset{\text{def}}{=} \widetilde{\Phi_{\overline{M}}^{\omega}}([0,\infty), x).$$

Notice that $\overline{A_x}$, the closure of A_x in the compactification of perspective, has only ω in its remainder. Then for each $i \in \mathbf{N}$ we let A_i denote $A_{\mathcal{O}(t_i)}$, so that $\left\{\overline{A_i}\right\}_{i\in\mathbf{N}} \to \{\omega\}$ in the Hausdorff metric. If $\tau(\phi)$ denotes the flow

$$\mathbf{R} \times \sum\nolimits_{\overline{M}} \to \sum\nolimits_{\overline{M}};\ \tau(\phi)(t,x) = \phi(t, x-\tau) + \tau$$

and if $\mathcal{O}'$ is a lifted $\tau(\phi)$−orbit of $\tau + x_0 \in \sum_{\overline{M}}$ in $\mathbf{R}^n \times \{c'\}$, then we have $(s_0, c) = \mathcal{O}(0) \in \mathbf{R}^n \times \{c\}$ and $(s_0', c') = \mathcal{O}'(0) \in \mathbf{R}^n \times \{c'\}$ satisfying

$$\begin{aligned} \pi_{\overline{M}}(s_0) + c &= x_0 \text{ and } \pi_{\overline{M}}(s_0') + c' = x_0 + \tau, \text{ implying} \\ \pi_{\overline{M}}(s_0' - s_0) &= c - c' + \tau, \end{aligned}$$

which in turn implies that if $\widetilde{\tau}(s,c) \overset{\text{def}}{=} (s + s_0' - s_0, c')$, then

$$\begin{array}{ccc} \mathbf{R}^n \times \{c\} & \xrightarrow{\widetilde{\tau}} & \mathbf{R}^n \times \{c'\} \\ (\pi_{\overline{M}} + c) \downarrow & & \downarrow (\pi_{\overline{M}} + c') \\ \sum_{\overline{M}} & \xrightarrow{+\tau} & \sum_{\overline{M}} \end{array}$$

is a commutative diagram of maps and $\widetilde{\tau}$ carries orbits of $\widetilde{\Phi^{\omega}_{\overline{M}}}$ to orbits of $\widetilde{\Phi^{\omega}_{\overline{M}}}$ and takes A_i in $\mathbf{R}^n \times \{c\}$ to a corresponding ray A_i' of a $\widetilde{\Phi^{\omega}_{\overline{M}}}-$ orbit in $\mathbf{R}^n \times \{c'\}$. The rays A_i' are translates of the rays A_i by $s_0' - s_0$ and this difference is negligible under the mapping h_n as $i \to \infty$. Again taking closures in the compactification of perspective, $\left\{\overline{A_i'}\right\}_{i \in \mathbf{N}} \to \{\omega\}$ in the Hausdorff metric, implying that

$$\{\mathcal{O}'(t_i)\}_{i \in \mathbf{N}} \to \omega.$$

Now the map $\delta(x) \overset{\text{def}}{=} h(x) - (\tau + x)$ is homotopic to the constant map $\sum_{\overline{M}} \to \{e_{\overline{M}}\}$, and so there is lift $\tilde{\delta}$ making the following diagram commute

$$\begin{array}{ccc} & & \mathbf{R}^n \\ & \widetilde{\delta} \nearrow & \downarrow \pi_{\overline{M}} \\ \sum_{\overline{M}} & \overset{\delta}{\longrightarrow} & \sum_{\overline{M}} \end{array}.$$

Since $\sum_{\overline{M}}$ is compact, $\left\|\widetilde{\delta}(x)\right\|$ has a maximum value m. Also, with

$$\widetilde{h}(s,c) \overset{\text{def}}{=} \left(s + s_0' - s_0 + \widetilde{\delta}(\pi_{\overline{M}}(s) + c), c'\right) = \text{``}\left(\widetilde{\delta}(\pi_{\overline{M}}(s) + c), c'\right) + \widetilde{\tau}(s,c),\text{''}$$

the following diagram commutes

$$\begin{array}{ccc} \mathbf{R}^n \times \{c\} & \overset{\widetilde{h}}{\longrightarrow} & \mathbf{R}^n \times \{c'\} \\ (\pi_{\overline{M}} + c) \downarrow & & \downarrow (\pi_{\overline{M}} + c') \\ \sum_{\overline{M}} & \overset{h}{\longrightarrow} & \sum_{\overline{M}} \end{array}.$$

Thus, identifying $\mathbf{R}^n \times \{c'\}$ with $\mathbf{R}^n$, $\left\|\widetilde{h}(s,c) - \widetilde{\tau}(s,c)\right\| \le m$. For any preassigned number K we may choose N sufficiently large so that for all $i \ge N$ $\{\inf \|\widetilde{\tau}(a)\| \,|a \in A_i\} > K$ in $\mathbf{R}^n \times \{c'\}$. By the construction of h_n and the uniform bound $\left\|\widetilde{h}(s,c) - \widetilde{\tau}(s,c)\right\| \le m$, we conclude that $\left\{\overline{\widetilde{h}(A_i)}\right\}_{i \in \mathbf{N}} \to \{\omega\}$ in the compactification of perspective. The analysis is very similar to that of the torus case ([ABZ, Chapter 6,1.5]), where the difference function is not only uniformly bounded but also doubly periodic. Each $\widetilde{h}(A_i)$ contains a point $\widetilde{h}(\mathcal{O}(t_i)) = \mathcal{O}''(t_i'')$ in a lift $\mathcal{O}''$ of the $\psi-$ orbit of $h(x_0)$. While there is no reason to expect that $t_i'' = t_i$, we may conclude $\{t_i''\}_{i \in \mathbf{N}} \to \infty$ since h preserves the orientation of orbits and $\{t_i\}_{i \in \mathbf{N}} \to \infty$. Hence, ω is a point of perspective of ψ as well. The other set containment follows mutatis mutandis. $\square$

Note This implies that $\rho(\phi) = \rho(\psi)$. It follows by setting $\tau = e_{\overline{M}}$ in the above proof that if $\mathcal{O}$ is a lifted orbit having ω as a forward limit point, then any other lift of the same orbit also has ω as a forward limit point.

Lemma 6. *If $\alpha \in Aut\left(\sum_{\overline{M}}\right)$ and $\alpha : \phi \overset{top}{\approx} \psi$, then $\rho(\phi) = \rho(\psi)$.*

Proof Let $\omega \in S^{n-1}$, $\mathcal{O}$, $x_0 \in \sum_{\overline{M}}$ and $\{t_i\}_{i\in\mathbf{N}} \to \infty$ be as above. With $\omega' = \widehat{A(\alpha)}(\omega)$, α maps orbits of $\Phi^{\omega}_{\overline{M}}$ to orbits of $\Phi^{\omega'}_{\overline{M}}$ after (possibly) rescaling time (see [C2, 3.5]), and α lifts to an affine map $\mathbf{R}^n \times \{c\} \to \mathbf{R}^n \times \{c'\}$, so that the above argument for $\tau(\phi)$ can be slightly adapted and applied here to show that ω' is a point of perspective of ψ. Since $\overline{M}(\{\omega\}) = \overline{M}(\{\omega'\})$, we conclude that $\rho(\phi) \subset \rho(\psi)$. Similarly, with α^{-1} in place of α, $\rho(\phi) \supset \rho(\psi)$. $\square$

Theorem 7. *If the flows ϕ and ψ on $\sum_{\overline{M}}$ are topologically equivalent, then $\rho(\phi) = \rho(\psi)$.*

Proof Suppose that $h : \phi \overset{\text{top}}{\approx} \psi$ and let $\omega \in S^{n-1}$, $\mathcal{O}$, $x_0 \in \sum_{\overline{M}}$ and $\{t_i\}_{i\in\mathbf{N}} \to \infty$ be as above. With $\tau : \sum_{\overline{M}} \to \sum_{\overline{M}}$ denoting translation by $h(e_{\overline{M}})$, the homeomorphism $g = \tau^{-1} \circ h$ fixes the identity element $e_{\overline{M}}$. From [S] (or see [C2, 3.8]), it follows that there is a homotopy

$$H : [0,1] \times \sum\nolimits_{\overline{M}} \to \sum\nolimits_{\overline{M}}$$

from $g = H_0$ to an automorphism $\alpha = H_1$. Then

$$h' \overset{\text{def}}{=} h \circ \alpha^{-1} = \tau \circ g \circ \alpha^{-1}$$

is homotopic to τ via $\tau \circ H_t \circ \alpha^{-1}$, implying that $h = h' \circ \alpha$, where h' is homotopic to a translation. Then by the above lemmas it follows that $\omega' = \widehat{A(\alpha)}(\omega)$ is a point of perspective of ψ. Since $\overline{M}(\{\omega\}) = \overline{M}(\{\omega'\})$, we conclude that $\rho(\phi) \subset \rho(\psi)$ and similarly $\rho(\phi) \supset \rho(\psi)$. $\square$

Notice that if we replace $\{t_i\}_{i\in\mathbf{N}} \to \infty$ with $\{t_i\}_{i\in\mathbf{N}} \to \pm\infty$ in definition 3, we could obtain a similar invariant. Also, in analogy with the invariant introduced in [MZ], one could consider all limit points of sequences of the form $\{\mathcal{O}_{x_i}(t_i)\}_{i\in\mathbf{N}}$ where x_i is allowed to vary and $\{t_i\}_{i\in\mathbf{N}} \to \infty$.

4. Exponents and Other Invariants

We now compare $\rho(\phi)$ with the exponent group of an orbit $\phi_x : \mathbf{R} \to X$ of a flow ϕ as introduced in [C1]. The exponent group of

ϕ_x is the subgroup of $(\mathbf{R},+)$ given by

$$\left\{\alpha\in\mathbf{R}\mid\begin{array}{c}\{\pi^1(\alpha t_i)\}\text{ converges in }S^1\text{ for all sequences }\{t_i\}\\ \text{for which }\{\phi_x(t_i)\}\text{ converges in }X\end{array}\right\}.$$

Recall that a non-empty set $\mathcal{M}\subset X$ is a *minimal set* of the flow ϕ on X if and only if $\mathcal{M}$ is closed and invariant and has no proper non-empty subset which is also closed and invariant. In [C1, Theorem 5] the exponent group is shown to be the same for all the orbits of a minimal set of a flow, allowing one to associate a group $\mathcal{E}(\mathcal{M})$ with each minimal set $\mathcal{M}$. For example, an $\alpha-$irrational flow on the torus formed by taking the suspension of rotation by α on S^1 is minimal and its exponent group is $\{m+n\alpha|m,n\in\mathbf{Z}\}$. While this group is a C^0-conjugacy invariant [C1, Corollary 6], it is not invariant under time changes since there are irrational flows which are not conjugate to flows to which they are topologically equivalent (see, e.g., [NS, V.8.19]). Such time changed flows will have the same rotation classes as the original flows but different exponent groups. The exponent group is sensitive to changes in the ergodic properties of the flows (stability) which the rotation class does not detect. Which is more desirable depends on what one wishes to investigate: topological equivalence or ergodic properties.

If $h:X\rightarrow Y$ provides a continuous semiconjugacy between a flow ϕ on X and a flow ψ on Y, ϕ is said to be an *almost 1:1-extension* of ψ when there is some $x\in X$ with $h^{-1}(h(x))=\{x\}$, see, e.g., [V, Chapter IV,6.1]. The following theorem allows one to calculate $\mathcal{E}(\mathcal{M})$ for a large class of minimal sets. Recall that an orbit of a flow ϕ on a compact metric space X is *almost periodic* if and only if it is uniformly Lyapunov stable (i.e., equicontinuous: see, e.g., [NS, Chapter V.11.7]).

Theorem 8. *If the minimal flow ϕ on a compact space $\mathcal{M}$ is an almost 1:1-extension of the almost periodic flow ψ on Σ, then $\mathcal{E}(\mathcal{M})=\mathcal{E}(\Sigma)$.*

Proof This argument depends critically on the results and constructions to be found in [C1]. We assume that the flow ψ is represented as a linear flow on a $\kappa-$solenoid Σ with identity e. As ψ is invariant under translations, we may assume that the semiconjugacy $h:\mathcal{M}\rightarrow\Sigma$ is such that $h^{-1}(h(x))=h^{-1}(e)=\{x\}$ for some $x\in\mathcal{M}$. From [C1, Theorem 5] it follows that $\mathcal{E}(\mathcal{M})\supset\mathcal{E}(\Sigma)$. Generally, $\mathcal{E}(\mathcal{M})$ coincides with the exponent group of a maximal irrational linear flow μ on a topological group $\mathcal{G}$ semiconjugate to ϕ, and $\mathcal{E}(\mathcal{M})$ is naturally isomorphic with the dual group of the topological group supporting the flow μ, which is isomorphic to $\check{H}^1(\mathcal{G})$. If $g:\mathcal{M}\rightarrow\mathcal{G}$

provides a semiconjugacy of ϕ with μ, again we may assume that $g(x)$ is the identity of $\mathcal{G}$. If the containment $\mathcal{E}(\Sigma) \subset \mathcal{E}(\mathcal{M})$ is proper, the Pontryagin dual epimorphism $E : \mathcal{G} \to \Sigma$ is not an isomorphism. Then E also semiconjugates the flow μ with the flow ψ. Since the flows involved are minimal, two semiconjugacies coinciding at a point must be identical. From this it follows that $E \circ g = h$. But E is not 1:1 and g is surjective, contradicting the assumption that $h^{-1}(h(x)) = \{x\}$. We must therefore have $\mathcal{E}(\Sigma) = \mathcal{E}(\mathcal{M})$. □

Since the natural semiconjugacy of a Denjoy flow with the irrational flow from which it is constructed is an almost 1:1 extension, this allows us to calculate its exponent group easily. For the natural flow on an $\alpha-$ Denjoy minimal set $\mathbb{D}_\alpha$ (regardless of the number of components in the complement) the exponent group is $\{m + n\alpha | m, n \in \mathbf{Z}\}$. Translating Fokkink's classification of the Denjoy continua with connected complements [F], [BW], we have:

$$[\mathbb{D}_\alpha \text{ is homeomorphic to } \mathbb{D}_\beta]$$

$$\Updownarrow$$

$$[\mathcal{E}(\mathbb{D}_\alpha) = c \cdot \mathcal{E}(\mathbb{D}_\beta) \text{ for some constant } c]$$

$$\Updownarrow$$

$$\left[\overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \left(\frac{(1,\alpha)}{\|(1,\alpha)\|}\right) = \overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \left(\frac{(1,\beta)}{\|(1,\beta)\|}\right)\right]$$

which is equivalent to requiring that α and β have continued fraction expansions with a common tail [BW],[ABZ, Chapter 6.1.7], and so all of the invariants under consideration suffice to classify these minimal sets *topologically*. (Recall that

$$\left\{\widehat{A(\alpha)} | \alpha \in Aut\left(\mathbf{T}^2\right)\right\} = GL(2, \mathbf{Z}).)$$

It is then natural to wonder whether there is a 1-dimensional compact minimal flow having a given non-trivial countable subgroup $\mathcal{E} \subset (\mathbf{R}, +)$ as its exponent group. We now construct such a flow for subgroups with finite torsion-free rank (see [Fu]). First we find a maximal rationally independent subset

$$A = \{\alpha_i\}_{i=1}^n \subset \mathcal{E}$$

(n is the torsion-free rank of $\mathcal{E}$) with

$$\{\alpha_i\}_{i=1}^{n-1} \subset \mathbf{R} - \mathbf{Q} \text{ (it is possible that } A \subset \mathbf{R} - \mathbf{Q}).$$

If $n = 1$ we obtain the desired flow by a linear flow on a $1-$solenoid. When $n > 1$, for each $i \in \{1, .., n-1\}$ we construct a Denjoy homeomorphism h_i of S^1 with rotation number α_i. Then for each i

there is a monotone map $\sigma_i : S^1 \to S^1$ isotopic to id_{S^1} which provides a semiconjugacy between h_i and rotation by $\pi^1(\alpha_i)$ in S^1. Each such h_i has a minimal Cantor Set C_i. Then

$$C = \prod_{i=1}^{n-1} C_i$$

is yet another Cantor Set and

$$h\left((x_i)_{i=1}^{n-1}\right) \stackrel{\text{def}}{=} (h_i(x_i))_{i=1}^{n-1}$$

is a homeomorphism of $\mathbf{T}^{n-1}$ that has C as an invariant set since each C_i is h_i-invariant. Also

$$\sigma\left((x_i)_{i=1}^{n-1}\right) \stackrel{\text{def}}{=} (\sigma_i(x_i))_{i=1}^{n-1}$$

is isotopic to $id_{\mathbf{T}^{n-1}}$ and provides a semiconjugacy between h and R, rotation in $\mathbf{T}^{n-1}$ by $\left(\pi^1(\alpha_i)\right)_{i=1}^{n-1}$, which is minimal by the rational independence of the α_i. The semiconjugacy of $h|_C$ to R shows that $h|_C$ is minimal [V, Chapter IV.6.1(b)] and $h|_C$ is an almost 1:1 extension of R. The suspension of h will then be a flow on $\mathbf{T}^n$ having a one–dimensional minimal set having C as a $0-$ dimensional cross-section. By adjusting the time scale on this flow to have a return time to C of $1/\alpha_n$, we obtain a minimal flow ϕ on a one–dimensional subcontinuum of $\mathbf{T}^n$, and the orbits of this flow have as their exponent group $\langle A\rangle$, the subgroup of $(\mathbf{R}, +)$ generated by A since this flow is an almost 1:1 extension of the linear flow Φ^α on $\mathbf{T}^n$ having exponent group $\langle A\rangle$ one obtains by taking the suspension of R and rescaling time by a factor of $1/\alpha_n$.

By enumerating the elements of $\mathcal{E} - \langle A\rangle = \{g_1, g_2, ...\}$ (possibly empty), we obtain a direct limit representation of $\mathcal{E}$

$$\langle A\rangle \hookrightarrow \langle A \cup \{g_1\}\rangle \hookrightarrow \langle A \cup \{g_1, g_2\}\rangle \hookrightarrow \cdots \mathcal{E}$$

and dual to this direct sequence is an inverse sequence

$$\mathbf{T}^n \stackrel{E_1}{\twoheadleftarrow} \mathbf{T}^n \stackrel{E_2}{\twoheadleftarrow} \mathbf{T}^n \stackrel{E_3}{\twoheadleftarrow} \cdots \sum\nolimits_{\overline{M}}$$

where each E_i is an epimorphism and the inverse limit an $n-$solenoid. Pulling the flow ϕ back by the various E_i as described in [C3], we obtain a flow on $\sum_{\overline{M}}$. This flow has a one–dimensional invariant set $\mathcal{M}$ which is minimal [V, Chapter IV.6.1(b)] and is an almost 1:1 extension of the corresponding linear flow obtained by pulling back Φ^α to a linear flow on $\sum_{\overline{M}}$. The orbits of the flow on $\mathcal{M}$ have $\mathcal{E}$ as their exponent group since this is the exponent group of the corresponding linear flow on $\sum_{\overline{M}}$, see [C1, Section 4]. This construction in the case $n = 2$ is considered in detail in [C3] where

such minimal sets (called denjoids) are classified topologically. In the case of a general denjoid the exponent group is not finitely generated and has torsion-free rank 2.

These flows will have only one point of perspective:

$$\alpha = \frac{(\alpha_1, ..., \alpha_n)}{\|(\alpha_1, ..., \alpha_n)\|}$$

since the lifted flow lines in $\mathbf{R}^n \times \{c\}$ differ from a lifted irrational flow by a map (obtained from $\sigma(x) - x$) homotopic to a constant, implying the existence of a bounded $\tilde{\delta}$ as in 5 (see [ABZ, Chapter 6.1.7] for similar arguments). Once again we find that the exponent group and the rotation class give us the same information when analyzing these flows on a fixed $\sum_{\overline{M}}$ since the rotation classes of such flows are identical precisely when their exponent groups are multiples of each other. However, the exponent group is independent of the embedding in $\sum_{\overline{M}}$ and so can be used to compare flows on $\sum_{\overline{M}}$ and $\sum_{\overline{N}}$ quite easily. To translate information about the rotation class of a flow on $\sum_{\overline{M}}$ to information about the class of a flow on $\sum_{\overline{N}}$ when $\sum_{\overline{M}}$ and $\sum_{\overline{N}}$ are isomorphic, one can use a matrix representing an isomorphism $i : \sum_{\overline{M}} \to \sum_{\overline{N}}$, see [C2, Theorem 3.4].

We now compare the rotation class with the rotation sets as discussed in [FM]. There examples are discussed with rotation sets that are intervals. In particular, the example due to Katok of a time changed irrational flow with a singular point has an interval as its rotation set, but all the flow lines with unbounded lifted forward orbits would have the same point of perspective as the original irrational flow. Once again, which invariant is best depends on the goal of the investigation.

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